A Note on the Dimension of Banach Spaces

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개 요

이 논문의 목적은 Basis의 Cardinality가 무한할 때 Banach공간을 정의하기 위하여 Hamel basis를 적용해야 하는 이유를 분명히 하고 또한 무한차원의 Banach 공간 X의 임의의 차원은 연속체의 Cardinality c만큼 크다는 것을 증명하기 위한 것이다. 또한 이 논문의 III에서 보여 주고 있는 증명은 참고문헌 1. 에서는 suprimum norm을 가진 유한 수열 로된 L^∞공간을 통해 증명하도록 암시하고 있으나 이와는 관계없이 독자적임을 부언해 둔다.

INTRODUCTION

The purpose of this paper is to clarify why we should apply Hamel bssis in order to define the dimension of Banach spaces when some infinite cardinality of a basis is encountered and to prove the dimension of any infinite dimensional Banach space X is at least as great as the cardinality c of the continum. The proof we have presented in part III is quite an independent one comparing the suggestion given in 1. which is to try through the space L^∞ of bounded sequences with the suprimum norm.

I

A *Hamel basis* for a linear space L is defined to be a linearly independent subset H which spans L. Hence any element in L can be expressed in terms of a finite linear combination of the elements in H. Since we have as a fundamental fact that every linear space has a Hamel basis and as an easy consquence that any two Hamel basis of a linear space are in one-to-one

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correspondence, the cardinality of the set of a Hamel basis of a linear space L can be served as the (Hamel) dimension of L.

A Schauder basis for a linear metric space M is a sequence $\{a_n\}$ such that for any vector $x \varepsilon$ M there exists a unique sequence $\{k_n\}$ of scalars such that $\sum_{n=1}^{\infty} k_n a_n = x$. The difference in two conceptions, Hamel basis and Schauder basis, does not occur in any finite dimensional normed linear space. However in the infinite dimensional normed linear spaces, two conceptions become different. The Schauder basis allows infinite sums while in the Hamel basis only finite sums occur 2.

The following theorm will signify that why it is reasonable for us to admit Hamel basis instead Schauder basis to define the dimension of Banach spaces in general.

THEOREM For a normed linear space X to admit a countable Schauder basis, X must be separable in the strong topology.

PROOF. Let $x \varepsilon$ X, and $\{a_i\}$ be a countable Schauder basis. Then

$$\lim_{n \to \infty} \| x - \sum_{i=1}^{n} x_i a_i \| = 0.$$

that is, for arbitrary $\varepsilon > 0$, there is a number n_0 such that

$$\| x - \sum_{i=1}^{n_0} x_i a_i \| < \frac{\varepsilon}{2}$$

where x_i are scalars.

Now we consider a countable set of elements in X such that to each of it corresponds the sequence $\{r_i\}$ of rational scalars in its expression with respect to the basis $\{a_i\}$. For a suitable sequence $\{r_i\}$ we can have

$$\| x - \sum_{i=1}^{n_0} r_i a_i \| \leq \| x - \sum_{i=1}^{n_0} x_i a_i \| + \sum_{i=1}^{n_0} |x_i - r_i| \, \| a_i \|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This proves the theorem.

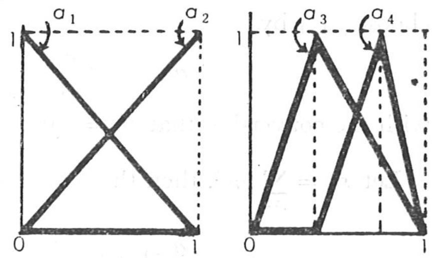
II

The property of completeness of normed linear spaces encounter with these two basis differently and hence we shall arrive at quite a strong restriction to the dimension of Banach space (part III).

We consider the Banach space C[0,1], the space of continuous functions on the closed unit interval I=[0,1] with the sup. norm, and we shall prove the following theorem for the comparison with Theorem in part III.

THEOREM C[0,1] has a countable Schauder basis.

PROOF. Let r_i, $i = 1,2,3,\cdots\cdots$ with $r_1 = 0$, $r_2 = 1$ be a dense subset of [0,1]. We define a countable basis (a_i) in C[0,1] as suggested in the figures.



Now let us define the sequence $\{k_i\}$ of the scalars for arbitrary given $f \varepsilon$ C[0,1] by

$$k_1 = f(r_1) = f(0)$$
$$k_2 = f(r_2) = f(1).$$

To go on inductively, we set $s_2 = k_1a_1 + k_2a_2$, and then define

$$k_3 = f(r_3) - s_2(r_3), \cdots\cdots$$

Thus for arbitrary n, we set $s_n = \sum_{i=1}^{n} k_i a_i$

Then

$$s_n(t) = f(t), \quad \text{for } t = r_1, r_2, \cdots, r_n$$

and $f = \sum_{i=1}^{\infty} k_i a_i.$

Let $g_i(t) = f(r_i)$, thene $g_i \in C[0,1]^*$, the duel space.

Suppose $f(t) = \sum_{i=1}^{\infty} k_i a_i(t) = \sum_{i=1}^{\infty} h_i a_i(t),$

then $\sum_{i=1}^{\infty} (k_i - h_i)\, a_i(t) = 0$ for all $t \epsilon [0, 1]$

$$0 = g_j\left(\sum_{i=1}^{\infty} (k_i - h_i)\, a_i\right) = \sum_{i=1}^{\infty} (k_i h_i)\, g_j(a_i) = k_j - h_j.$$

Hence $k_i = h_i$ for all $i = 1,2,\ldots\ldots,$

This proves the uniqueness and completes the proof.

III

The purpose of this part is to see how much different phenomenon would appear concerning with the Hamel basis for the space C[0,1]. However we shall prove it as more general case in an arbitrary normal linear compete space X.

THEOREM If X is an infinite dimensional Banach space. Then dim $X \geq c$, the cardinality of the continum.

PROOF. We have to show that there always exists an element in X that does not belong to a subspace generated by any countable linearly independent vectors in X.

Let $\{a_n\}_{x \varepsilon w}$ be a sequence of linearly independent elements in X. We define inductively a sequence $\{\mu_n\}$ of positive real numbers as follows:

If $d_n = d(\mu_n a_n, V_{n-1})$, where $V_{n-1} = [a_1, \cdots, a_{n-1}]$, the subspace generated by $a_1, \cdots, a_{n-1}$ and d is the metric induced from the norm, then we choose μ_{n+1} by

$$\mu_{n+1} \| a_{n+1} \| \leq \frac{d_n}{3}$$

with the conversion that $V_0 = \{0\}$, and $\mu_1 = 1$.

Let $s_n = \sum_{i=1}^{n} \mu_i a_i$, then the sequence $\{s_n\}$ of series absolutely converges. In fact

$$\| \mu_{n+1} a_{n+1} \| \leq \frac{d_n}{3}$$

and on the other hand,

$$d_n = d(\mu_n a_n, V_{n-1}) \leq d(\mu_n a_n, 0) = \| \mu_n a_n \| \leq \frac{d_{n-1}}{3},$$

whence $$d_n \leq \frac{d_{n-1}}{3} \leq \frac{d_{n-i}}{3^i} \leq \frac{d_1}{3^{n-1}} = \frac{\| a_1 \|}{3^{n-1}}.$$

For arbitrary n,

$$\sum_{i=1}^{n} d_i \leq \sum_{i=1}^{n} \frac{\| a_1 \|}{3^{i-1}} = \| a_1 \| \left(1 + \frac{1}{3} + \cdots + \frac{1}{3^{n-1}}\right),$$

and hence

$$\sum_{i=1}^{\infty} d_i = \| a_1 \| \frac{1}{1-\frac{1}{3}} = \frac{2}{3} \| a_1 \|$$

This proves $\{s_n\}$ absolutely converges, and therefore $\{s_n\}_{n \varepsilon w}$ is a Cauchy sequence in X. Hence it has a limit point, say

$$x_0 = \lim_{n \to \infty} s_n = \sum_{i=1}^{\infty} \mu_i a_i.$$

We shall show that x_0 does not belong to V, the sequence of X generated by the set $\{a_i\}_{i \varepsilon w}$.

Suppose $x_0 \varepsilon V$, and for arbitrary small number $\varepsilon > 0$. We let $U_\varepsilon = \{x \in X : \| x - x_0 \| < \varepsilon\}$ Then there exists an integer N such that for arbitrary given $n > N$,

$$s_n = \sum_{i=1}^{n} \mu_i a_i \varepsilon V_n \text{ and } s_n \in U_\varepsilon.$$

On the other hand there exists an integer m such that

$$x_0 = \sum_{i=1}^{m} \xi_i a_i \text{ (uniquely expressed), where } \xi_i \text{ are scalars.}$$

Now

$$\| x_0 - s_n \| = \| \sum_{i=1}^{m} \xi_i a_i - \sum_{i=1}^{n} \mu_i a_i \|$$
$$= \| \sum_{i=1}^{m} (\xi_i - \mu_i) a_i - \sum_{i=m+1}^{n} \mu_i a_i \|$$

If we set

$y_0 = \sum_{i=1}^{m} (\xi_i - \mu_i) a_i$, than $y_0 \in V_m$, and hence

$$(*) \quad \| x_0 - s_n \| = \| y_0 - \sum_{i=m+1}^{n} \mu_i a_i \| = \| y_0 - \mu_{m+1} a_{m+1} - \sum_{i=m+2}^{n} \mu_i a_i \|$$

$$\geq \Big| \| y_0 - \mu_{m+1} a_{m+1} \| - \| \sum_{i=m+2}^{n} \mu_i a_i \| \Big|$$

Since $y_0 \in V_m$, according to the sequence $\{\mu_i\}$ we have constructed,

$$\| y_0 - \mu_{m+1} a_{m+1} \| \geqq d(\mu_{m+1} a_{m+1}, V_m) = d_{m+1},$$

and

$$\| \sum_{i=m+2}^{n} \mu_i a_i \| \leq \sum_{i=m+2}^{n} \| \mu_i a_i \| \leq \frac{d_{m+1}}{3} + \frac{d_{m+2}}{3} + \cdots + \frac{d_{n-1}}{3}$$

$$\leq \frac{d_{m+1}}{3} + \frac{d_{m+1}}{3^2} + \cdots + \frac{d_{m+1}}{3^{n-m+1}}$$

since

$$d_n \leq \frac{1}{3^i} d_{n-i}.$$

Thus

$$\| \sum_{i=1}^{n} \mu_i a_i \| \leq \frac{d_{m+1}}{2} \cdot \frac{3^{n-m+1} - 1}{3^{n-m+1}} < \frac{d_{m+1}}{2}$$

Combining this inequality together with $(*)$

$$\| x_0 - s_n \| \geq d_{m+1} - \frac{d_{m+1}}{2} = \frac{d_{m+1}}{2}$$

This contradicts the fact that we have chosen sufficiently large N so that $\| x_0 - s_n \| < \varepsilon$ for arbitrary small positive number ε. The refore x_0 must not belong to V. This completes the proof.

References

1. E. Taylor, Introduction to Functional Analysis, John wiley & Son, New York, 1958.
2. A. Wilansky, Functional Analysis, Blaisdell publishing Co., New York, 1964.

Spring, 1971.
Taejon, Korea.