A Note on Semi-Developable Spaces

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개　요

본 논문의 중요 결과는 open semi-developable 부분 공간들의 합은 만약 그합공간이 F_σ-screenable 이라면 semi-developable 이 된다. 그리고 semi-development 를 갖고 있는 Lindelof 공간의 임의 부분 공간도 Lindelof 공간이 되고, point-finite semi-developable 공간에서는 Lindelof 와 separability 는 같다.

A space X is developable if there is a sequence $\Delta = \{g_n | n \in \mathcal{N}\}$ of open covers of X such that $\{St(x, g_n) | n \in \mathcal{N}\}$ is a local base at x, for each $x \in X$. Recently Charles C. Alexander [1] introduced a semi-developable space as a generalization of the developable space and showed some properties of the space.

The principal results of this paper are as follows. The union X of open semi-developable subspaces is semi-developable if X is F_σ-screenable. A Lindelöf space with a semi-development is hereditarily Lindelöf. In a point-finite semi-developable space, the Lindelöf property is equivalent to separability.

The notation and terminology used in this paper are to follow those of J. C. kelley [4] mainly. N is the set of positive integers.

DEFINITION 1. [1] A *semi-development* for a space X is a sequence $\{g_n | n \in \mathcal{N}\}$ of (not necessarily open) covers of X such that $\{St(x, g_n) | n \in \mathcal{N}\}$ is a neighborhood base at x, for each x $\in$ X.

A space is semi-developable space if and only if there exists a semi-development for the space.

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THEOREM 2. A Lindelöf space with a semi-development is hereditarily Lindelöf.

PROOF. Let X be a Lindelöf semi-developable space. A semi-developable space is hereditary (p. 278 in [1]). Since a Lindelöf space is hereditarily Lindelöf if and only if each open subspace is Lindelöf (P. 144 in [2]). Let U be an open set in X and $\triangle=\{g_n \mid n \in \mathcal{N}\}$ be a semi-development for X. If we take $U_n=\overline{[\mathrm{St}(U', g_n)]'}$, then $U=\bigcup_{n=1}^{\infty} U_n$.

For, let $y \in \bigcup_{n=1}^{\infty} U_n$, there is an integer m such that $y \in U_m$ (i.e., $y \in \overline{[\mathrm{St}(U', g_m)]'}$). Suppose that $y \notin$ U, then we have St $(y, g_m) \subset$ St (U', g_m). Therefore, we obtain St $(y, g_m) \cap [\mathrm{St}(U', g_m)]'=\phi$. Thus, $y \notin \overline{[\mathrm{St}(U', g_m)]'}$. This is contradict to $y \in \overline{[\mathrm{St}(U', g_m)'}]$.

For each $y \in$ U, there is an integer m such that St (y, g_m) $\subset$ U. Therefore we have St $(y, g_m) \cap$ U$'=\phi$. For such m, we obtain $y \notin$ St(U$'$, g_m). Thus, $y \in \overline{[\mathrm{St}(U', g_m)]'}=U_m$.

Since each Un is closed, hence Un is Lindelöf. Thus each open subspace is Lindelöf. Consequently X is hereditarily Lindelöf.

With the aid of Proposition 1.9 of [1] we have the following.

COROLLARY 3. Every Lindelöf semi-developable space is hereditarily separable.

A topological space is F_σ-*screenable* if every open cover has a σ-discrete closed refinement which covers the pace [3]

THEOREM 4. The union X of open semi-developable subspaces is semi-developable if X is F_σ-screenable.

PROOF. Let $X=\bigcup_{\alpha\in\mathscr{A}} U_\alpha$, where U_α be an open semi-developable subspaces with a semi-development $\triangle_\alpha=\{g_n[\alpha] \mid n \in \mathcal{N}\}$. Since X is F_α-screenable, there is a σ-discrete closed refinement $\mathscr{L}=\bigcup_{n=1}^{\infty}\mathscr{L}_n$ of $\{U_\alpha \mid \alpha\in\mathscr{A}\}$. For each $B\in\mathscr{L}_n$, there is a fixed element $\alpha(B)\in\mathscr{A}$ such that $B \subset U_{\alpha(B)}$.

Let $U_n(B)=X-\bigcup\{B^* \mid B^*\in\mathscr{L}_n, B^*\neq B\}$, $\mathscr{U}_{n,m}(B)=\{U_n(B)\cap G \mid G \in g_m[\alpha(B)]\}$ and $\mathscr{U}_{n,m}=\{U \mid U \in \mathscr{U}_{n,m}(B), B \in \mathscr{L}_n\} \cup \{Q_n\}$ where $Q_n=X-\bigcup\{B \mid B \in \mathscr{L}_n\}$.

Then $U_{n,m}$ is a cover of X for each $n, m \in \mathcal{N}$ and we show that $\{\mathscr{U}_{n,m} \in \mathcal{N}\}$ is a semi-development for X.

If z $\in$ X there is an integer $n \in \mathcal{N}$ with some B $\in \mathscr{L}_n$ such that z $\in B$. Consequently if O is any open set containing z there is some $m \in \mathcal{N}$ such that z $\in$ Int St (z, $g_m[\alpha(B)]$) $\subset$ St (z, $g_m[\alpha(B)]$ $\subset$ O $\cap U_{\alpha(B)}$. By construction z is not contained in any element of

$\mathscr{U}_{n,m}(B^*)$ for any $B^* \in \mathscr{L}_n$ such that $B_* \neq B$.

Since Int St $(z, \mathscr{U}_{n,m}) =$ Int St $(z, \mathscr{U}_{n,m}[B])$

$= \text{Int}\,(U_n[B] \cap \text{St}\,(z, g_m[\alpha(B)])$

$= U_n(B) \cap \text{Int St}\,(z.\ g_m[\alpha(B)]) \ni z.$

Thus $z \in \text{Int St}\,(z, \mathscr{U}_{n,m}) \subset \text{St}(z, \mathscr{U}_{n,m}) = \text{st}\,(z, \mathscr{U}_{n,m}[B]) \subset \text{st}\,(z, g_m[\alpha(B)]) \subset O$.

For each $k, l \in N$, we have $z \in \text{Int St}\,(z, U_{k,l})$. If there is some B such that $z \in B \in \mathscr{L}_n$, then we obtain $z \in \text{Int St}\,(z, \mathscr{U}_{k,l})$ by the abvoe way. If there is not a B such that $z \in B \in \mathscr{L}_n$, then $z \in Q_n$. Since Q_n is an open set, thus we have $z \in \text{Int St}\,(z, \mathscr{U}_{k,l})$. Hence $\{\text{St}\,(z, \mathscr{U}_{k,l}) \mid k, l \in N\}$ is a neighborhood base at z, for each $z \in X$.

DEFINITION 5. [3] A totological space X is a *semi-stratifiable* space if, to each open set $U \subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

(a) $\bigcup_{n=1}^{\infty} U_n = U$

(b) $U_n \subset V_n$ whenever $U \subset V$.

With the aid of Theorem 1.3 of [1] and Corollary 1.4 of [3], we have the following relationship between semi-developable spaces and semi-stratifiable spaces.

LEMMA 6. A space is semi-developable T_0 -space if and only if it is a first countable semi-stratifiable T_1-space.

A topological space is x_1-compact if every uncountable subset has a limit point. By applying the above Lemma and Theorem 2.8 of [3], we obtain the following theorem:

THEOREM 7. In a semi-developable T_0-space, the following are equivalent (1) X is Lindelöf, (2) X is hereditarily separable, and (3) X is x_1,–compact.

Alexander [1] showed that a separable space with a point-finite semi-development is hereditarily separable.

COROLLARY 8. In a point -finite semi-developable T_0-space, the Lindelöf property is equivalent to separability.

References

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