A Note on Semi-Stratifiable Spaces

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개 요

이 논문에서는 semi-stratifiable 공간의 몇가지 성질을 다루었다.

(1) locally finite 인 closed semi-stratifiable 공간들의 합은 semi—stratifiable 공간이 된다.

(2) p 가 공간 X 에서 공간 Y 로 가는 연속 개사상이고 X 는 p-saturate, Y 는 semi-stratifiable 공간이라면 X 도 또한 semi-stratifiable 공간이다.

(3) 만약 공간 X 가 semi-stratifiable 이라면 cone TX 도 또한 semi-stratifiable 이다.

Creede has been studied on semi-stratifiable spaces. He showed that the properties of the spaces.

We will show that some another properties of the semi-stratifiable spaces: (a) the union of closed semi-stratifiable spaces with locally finite is semi-stratifiable. (b)the attaching space of two semi-stratifiable spaces is semi-stratifiable. (c) p:X→Y is continuous open mapping, X is p-saturate and Y is semi-stratifiable, then X is semi-stratifiable. (d) if X is semi-stratifiable, then a cone TX is semi-stratifiable.

DEFINITION 1. [3] A topological space X is a *semi-stratifiable* space if, to each open set $U \subset X$, one can assign a sequence $\{U_n \mid n \in N\}$ of closed subsets of X such that

(a) $\bigcup_{n=1}^{\infty} U_n = U$

(b) $U_n \subset V_n$ whenever $U \subset V$

A correspondence $U \to \{U_n\}_{n=1}^{\infty}$ is a semi-stratification for the space X whenever it satisfies the conditions of Definition 1.

Let X,Y be two disjoint spaces, $A \subset X$ a closed subset, f: A→Y continuous. In X+Y, generate an equivalence relation R by $a \sim f(a)$ for each $a \in A$. The quotient space (X+Y)/

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R is called the attaching space. Denote by $X \cup_f Y$. [4]

A space X is p-saturated if each $A \subset X$, $p^{-1}p(A)=A$. For any space X, the cone TX over X is the quotiont space $X \times I/R$, where R is the equivalence relation $(x, 1) \sim (x', 1)$ for all $x, x' \in X$

LEMAN 2. (Creede) If Y is a closed subspace of a semi-stratifiable space X and $U \to \{U_n\}_{n=1}^{\infty}$ is a semi-stratification for Y, then there is a semi-stratification $V \to \{V_n\}_{n=1}^{\infty}$ for X such that $(V \cap Y)_n = (V_n \cap Y)$.

THEOREM 3. The union of closed semi-stratifiable subspaces of X with locally finite is semi-stratifiable.

PROOF Let $\{Y_\alpha | \alpha \in \mathscr{A}\}$ be a family of locally finite closed semi-stratifiable subspaces of X. Then $Y = \bigcup_{\alpha \in \mathscr{A}} Y_\alpha$ is closed in X. For $\{Y_\alpha | \alpha \in \mathscr{A}\}$ is locally finite.

Let O be an open set in Y.

a) $O \subset Y_\alpha - \bigcup_{\beta \neq \alpha} Y_\beta$, then we take $O \to \{O_n\}_{n=1}^{\infty}$ for Y_α.

b) $O \cap Y_\alpha \neq \phi$ for all $\alpha \in \mathscr{A}$.

In this case (b), if we apply the lemma 2 with respact to the common subspace and property of locally finite, the following statement is true; Let $O_\alpha = O \cap Y_\alpha$, then we take $O_\alpha \to \{O_{\alpha n}\}_{n=1}^{\infty}$ for Y_α. Then $O \longrightarrow \{O_n\}_{n=1}^{\infty}$ is a semi-stratification bor Y where $O = \bigcup_{\alpha \in \mathscr{A}} O_\alpha$, $On = \bigcup_{\alpha \in \mathscr{A}} O_{\alpha n}$.

For, O_n is closed in Y since $\{O_{\alpha n}\}_{\alpha \in \mathscr{A}}$ is locally finite.

(1) $\bigcup_{n=1}^{\infty} O_n = O$ (2) $O_n \subset O_n'$ whenever $O \subset O'$.

Creede [3] proved that the closed image of a semi-stratifiable space is semi-stratifiable. This result does not remain true if closed is replaced by open. But we have the following Theorem with respect to the open map.

THEOREM 4. Let $p: X \to Y$ be a continuous open mapping, if X is p-saturate, Y is semi-stratifiable, then X is semi-stratifiable.

PROOF. Let U be an open set in X. Then p(U) is open in Y. Hence we take $p(U) \to \{[p(U)]_n\}_{n=1}^{\infty}$ for Y. U is p-saturate, $U = p^{-1}p(U)$. Thus we have $U \to \{p^{-1}([p(U)]_n)\}_{n=1}^{\infty}$

is a semi-stratification for U in X. For, (1) $\bigcup_{n=1}^{\infty} p^{-1}([p(U)]_n) = p^{-1}(\bigcup_{n=1}^{\infty} [p(U)]_n)$ $=p^{-1}p(U)=U$ where $p^{-1}([p(U)]_n)$ is closed in X. (2) $p^{-1}([p(U)]_n) \subset p^{-1}([p(V)]_n)$ whenever $U \subset V$.

THEOREM 5. Let X be semi-stratifiable, then TX is semi-stratifiable.

PROOF. Let I ($\subset$R) be semi-stratifialbe. By theorem 2.1 of [3] X x I is semi-stratifiable. Since X x 1 is a closed subset of XxI, p: XxI $\rightarrow$ XxI/Xx1 is a continuous closed onto mapping. Hence the cone TX is semi-stratifiable.

Creede [3] Showed that a Moore space (regular developable space) is semi-stratifiable. We have that a developable space (not necessarily regular) is semi-stratifiable.

THEOREM 6. Every developable space is semi-stratifiable.

PROOF. Let X be a developable space and $\Delta=\{g_n | n \in N\}$ be a development for X. For each n and each open set $U \subset X$, we take $U_n=[St(U', g_n)]'$. Then the correspondence $U \rightarrow \{U_n | n \in N\}$ is a semi-startification for X.

For, (a) $U=\bigcup_{n=1}^{\infty} Un$: Let $y \in \bigcup_{n=1}^{\infty} Un$, there is an integer m such that $y \in Um$(i.e., $y \in$ $[St(U', g_m)]'$). Suppose that $y \notin U$, then we have St $(y, g_m) \subset$ St (U', g_m). Therefore, we obtain St $(y, g_m) \cap$ [St $(U', g_m)]'=\phi$. Thus, $y \notin [St(U', g_m)]'$. This is contradict to $y \in [St(U', g_m)]'$.

For each $y \in U$, there is an integer m such that St $(y, g_m) \subset U$, Therefore we have St $(y, g_m) \cap U'=\phi$. For such m, we obtain $y \notin S_t (U', g_m)$. Thus, we have $y \in$ [St $(U', g_m)]'$ (i.e., $y \in U_m$).

(b) If U. V be open sets in X such that $U \subset V$, then we have St $(U', g_n) \supset$ St (V', g_n) for each n. Hence we obtain $[St(U', g_n)]' \subset$ [St $(V', g_n)]'$.

THEOREM 7. In a semi-stratifiable space X, we have the following:

(1) If X is normal, then X is perfectly normal.

(2) If X is Lindelöf, then X is hereditarily Lindelöf.

(3) If X is paracompact, then X is hereditarily paracompact.

PROOF. (1) is trivial

(2) A semi-stratifiable space has hereditary property. Since a Lindelof (paracompact) space is hereditarily Lindelof (paracompact) if and only if each open subspace is

Lindelof (paracompact)([2], [4]). Let U be an open set in X, then $U=\bigcup_{n=1}^{\infty}U_n$ where U_n is closed in X. Hence U is Lindelof. Consequentty, X is hereditarily Lindelof.

(3) If an open subset U in X is a F_σ-set, then U is paracompact. Hence it is trivial.

Using a proof analogous to one given by Borges for theorem 6.2 of [1], the following Theorem may be proved.

THEOREM 6. Let X and Y be semi-stratifiable, A a closed subset of X and f: A→Y a continuous function. Then $X \cup_f Y$ is semi-stratifiable.

References

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